# A GENERAL THEOREM RELATING TO TRANSVERSALS, AND ITS CONSEQUENCES.

By Mr. A. L. Candy, Lincoln, Nebraska.

Through the extremities of two fixed chords of a given circle two intersecting lines are drawn, and upon the two fixed chords circles are described passing through the intersection of this second pair of lines. Also, any right line is drawn through this same point of intersection. It is required to find the relation of the segments intercepted upon this transversal by the three circles and the two fixed chords; all segments being measured from the intersection of the second pair of lines.

Consider first the four segments determined by the given circle and the two fixed chords, under the following cases :*—

Case I. When the point of intersection is within the given circle and the transversal cuts the fixed chords within the given circle.

Let $CD$ and $C'D'$ be the two fixed chords, and let $O$ be the intersection of $C'D$ and $CD'$. Let any transversal, drawn through $O$, intersect the given circle in $A$ and $B$, $CD$ in $E_1$, and $C'D'$ in $E_2$, then $OA$, $OB$, $OE_1$, and $OE_2$ are the segments to be considered.

Let $A_1$, $A_2$, $A_3$, $A_4 \equiv$ the areas of the triangles $E_1OC$, $E_2OC'$, $E_1OD$, and $E_2OD'$, respectively.

Let $OB = m$, $OA = n$, $OE_2 = p$, $OE_1 = q$, $OC' = a$, $OD = b$, $OC = c$, $OD' = d$, $E_1C = x$, $E_1D = y$, $E_2C' = v$, and $E_2D' = z$.

Then,

$$\frac{A_1}{A_2} = \frac{cx}{av}, \quad (1) \qquad \frac{A_3}{A_4} = \frac{by}{dz}, \quad (2)$$

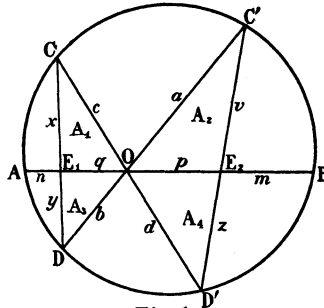$$\frac{A_1}{A_4} = \frac{cq}{dp}, \quad (3) \qquad \frac{A_3}{A_2} = \frac{bq}{ap}, \quad (4)$$



Fig. 1.

since $\angle C = \angle C'$, $\angle D = \angle D'$, etc.;

---

$\therefore$ 
$$\frac{A_1A_3}{A_2A_4} = \frac{bcq^2}{adp^2} = \frac{bcxy}{advz}, \tag{5}$$

whence

$$\frac{q^2}{p^2} = \frac{xy}{vz} = \frac{AE_1 \cdot E_1B}{AE_2 \cdot E_2B} = \frac{(n-q)(m+q)}{(n+p)(m-p)}, \tag{6}$$

$$\frac{q^2}{p^2} = \frac{mn - q(m-n) - q^2}{mn + p(m-n) - p^2}; \tag{7}$$

$\therefore$ 
$$mn(p-q) = pq(m-n), \tag{8}$$

by simplification, or

$$\frac{mn}{pq} = \frac{m-n}{p-q}. \tag{9}$$

$\therefore$ The products of the segments intercepted by the circle and the two chords, respectively, are proportional to their differences (or sums).

Since, by hypothesis, $AB$ is any chord through $O$, it can be so drawn that $O$ shall be its middle point. Then, $m - n = 0$, and from (8) $mn(p-q) = 0$; and therefore, since $mn \gtrless 0$,

$$p = q.* \tag{10}$$

CASE II. When $O$ is within the given circle and $E_1$ and $E_2$ are without.
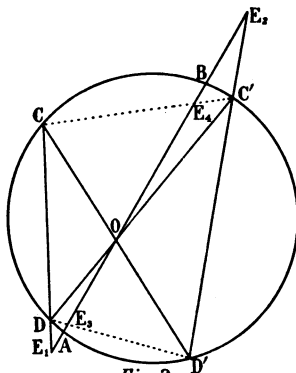


*Fig.2.*

The angles $C$ and $C'$, $D$ and $D'$ are now supplementary, and equations (1) ... (5) are still satisfied, but (6) becomes

$$\frac{q^2}{p^2} = \frac{(q-n)(m+q)}{(n+p)(p-m)}, \tag{11}$$

which again reduces to (8), since the signs of two factors are changed in the second fraction.

---

* See Stewart's Geom., p. 288, Ex. 29. It was this exercise that suggested this investigation; and all the following results will be found to be consequences of this simple relation in (8).

Next draw $CC'$ and $DD'$ intersecting $AB$ in $E_4$ and $E_3$, respectively; let $OE_4 = r$, $OE_3 = s$; then, as in Case I,

$$mn\,(r - s) = rs\,(m - n)\,;\qquad(12)$$

$$\therefore\qquad\frac{p - q}{r - s} = \frac{pq}{rs}\,,\qquad(13)$$

from (8) and (12).

∴ The segments intercepted by opposite pairs of chords joining the same points, possess the same relation as the segments made by the circle and one pair of opposite chords, and when any pair are equal all three pairs are equal.

When $AB$ meets $CD$ and $C'D'$ (produced) in the same direction from $O$ equation (8) becomes

$$mn\,(p + q) = pq\,(m - n)\,,\qquad(14)$$

which is just as it should be if we agree that direction shall determine the algebraic sign, for then $p$ and $q$ have the same direction and must be regarded as having the same signs.

CASE III. When $O$ is without the circle and $E_1$ and $E_2$ within.

The proof in this case is so similar to the first two that it is not necessary to produce it. However, it can be seen from Fig. 3 that $m$, $n$, $p$ and $q$ all extend in the same direction from $O$, and therefore $m$ and $n$ as well as $p$ and
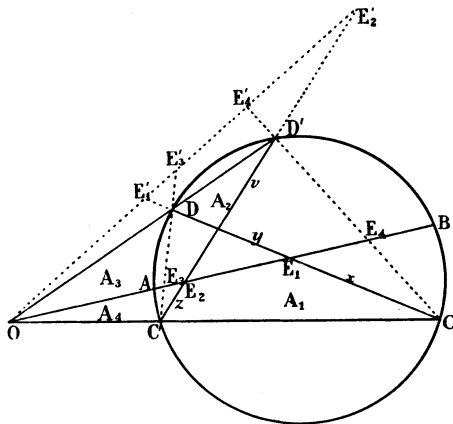


*Fig. 3.*

$q$ must be regarded as having the same sign, in order to make the preceding general statement agree with the result of the proof which gives

$$mn\,(p + q) = pq\,(m + n)\,.\qquad(15)$$

CASE IV. When $O$ is without the circle, and $CD$, $C'D'$ intersect within the circle.

Using the same triangles as in Case I, it is easily seen that

$$\frac{A_1 A_3}{A_2 A_4} = \frac{bcq^2}{adp^2} = \frac{bcxy}{advz},\qquad(16)$$

since $\angle C = \angle C'$, $\angle D = \angle D'$, and angles at $O$ are common.

$$\therefore\qquad \frac{q^2}{p^2} = \frac{xy}{vz} = \frac{(q - n)\,(m - q)}{(p - n)\,(m - p)},\qquad(17)$$

which gives as in Case III

$$mn\,(p + q) = pq\,(m + n).\qquad(18)$$

Now let $C'D$ and $CD'$ be drawn intersecting $AB$ in $E_3$ and $E_4$ respectively; let $OE_4 \equiv r$, $OE_3 \equiv s$; then substituting in (18) gives, by Case III,

$$mn\,(r + s) = rs\,(m + n);\qquad(19)$$

and dividing (18) by (19) gives

$$\frac{p + q}{r + s} = \frac{pq}{rs},\qquad(20)$$

an equation that is independent of $m$ and $n$.

This result is analogous to (13) but the interpretation is more inclusive. It shows that the relation between the segments formed by the two pairs of lines remains the same when the transversal does *not* intersect the given circle, as shown by $OE_1'E_3'$, etc., Fig. 3. We shall have occasion to refer to this equation again. It is well to notice in passing that when $O$ is on the given circle, either $m$ and $p$ or $n$ and $q$ both vanish, and (8) and (20) are verified.

It has now been shown that, having proper regard to signs, equations (8), (13), and (20) are true under all possible conditions, i. e. they are identical.

Next in order to determine the relation between the segments intercepted by the given circle and the two circles described upon the two fixed chords $CD$, $C'D'$, and passing through $O$, the process of inversion will be used.

Inverting Fig. 1, using $O$ as the centre of inversion gives Fig. 4, in which the inverse points are designated by the same letters; the lines $CD$ and $C'D'$ inverting into the circles $COD$ and $C'OD'$, respectively, lines through $O$ remaining unchanged.

Let $OB \equiv m'$, $OA \equiv n'$, $OE_2 \equiv p'$, $OE_1 \equiv q'$; and compare these with $m$, $n$, $p$, $q$ of Fig. 1; then

$$m = \frac{1}{m'},\qquad n = \frac{1}{n'},\qquad p = \frac{1}{p'},\qquad q = \frac{1}{q'};$$

and from (8)

$$\frac{1}{m'n'}\left[\frac{q'-p'}{p'q'}\right] = \frac{1}{p'q'}\left[\frac{n'-m'}{m'n'}\right];$$ (21)

$$\therefore \qquad q' - p' = n' - m'.$$ (22)

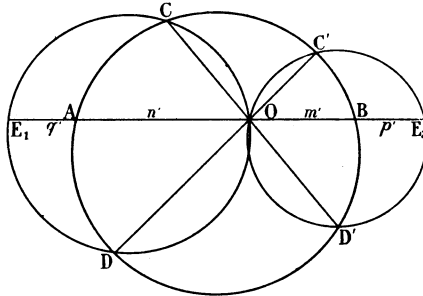∴ The difference of the segments intercepted by the two circles is equal



Fig. 4.

to the difference of the segments intercepted by the given circle.

Applying the same method to Fig. 2, using a similar notation, and substituting in (8) and (12), give

$$n' - m' = q' - p' = s' - r'.$$ (23)

∴ If circles be passed through $COC'$ and $DOD'$, $COD$ and $C'OD'$, respectively, the differences of the segments determined by these *two pairs* of circles on any transversal is always the same, and equal to the difference of the intercepts of the given circle.

It is obvious that if the same process be applied to Fig. 3, and the results substituted in (18) and (20) we should have

$$m' + n' = q' + p' \quad \text{and} \quad p' + q' = r' + s',$$ (24)

respectively.

∴ When $O$ is without the given circle (regarding the algebraic signs as before) the sums of the segments made by the two pairs of opposite circles are always equal whether the transversal cuts the given circle or not.

Now when $O$ becomes the middle point of $AB$,

$$n' = m', \quad q' = p', \quad s' = r',$$ (24a)

from (23), and also

$$n = m, \quad q = p, \quad s = r.$$

∴ When the segments intercepted by the given circle are equal, both pairs of opposite chords and both pairs of opposite circles described upon them intercept equal segments also.

Although these results have been proved by inversion it is clear that they are perfectly general, and hence we may drop the primes from $m$ and $n$.

So far we have discussed the two parts of this problem separately. Having found these separate relations, we shall now proceed to consider the complete solution.

Since the equations connecting the segments made by opposite chords and the given circle, and also those connecting segments made by the given
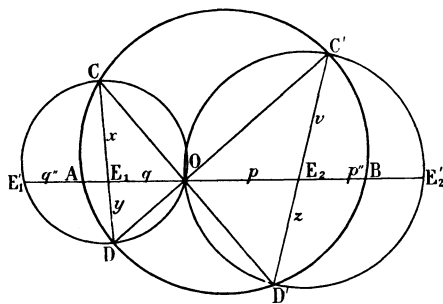


*Fig. 5.*

circle and a pair of circles upon these chords, have been shown to be identical under all possible conditions, it is sufficient now to consider only the simplest case, and we shall use the first.

Let Fig. 5 be the same as Fig. 1 with circles drawn through $COD$ and $C'OD'$ respectively ; let the notation be the same, and in addition let $OE_2' = p'$, $OE_1' = q'$, $E_2E_2' = p''$, $E_1E_1' = q''$ ; then

$$\frac{xy}{vz} = \frac{qq''}{pp''}. \tag{25}$$

On making the figures for all the different cases the truth of this equation also is at once perceived to be general.

Now from the first two fractions of (6) (and this equation is also found to be common to the proof of all the four cases), and from (25), we obtain

$$\frac{q^2}{p^2} = \frac{qq''}{pp''}, \tag{26}$$

whence $\dfrac{q}{p} = \dfrac{q''}{p''}$ , or

$$\therefore \qquad p : p'' :: q : q''. \tag{27}$$

By composition, remembering that $p + p'' = p'$ and $q + q'' = q'$, we get

$$p : p' :: q : q'. \tag{28}$$

From these two proportions the following general statements may now be made :—

Segments intercepted by two circles are divided proportionally by the chords upon which the circles are drawn.

Segments made by two opposite chords and circles described upon these chords are also proportional.

When $O$ is on the given circle, say at $A$, the circle $COD$ reduces to a point, and $q$, $q'$ are both equal to zero, and from (28) $\dfrac{p}{p'}$ takes the form $\dfrac{0}{0}$ and may have any value between zero and unity.

Now let

$$\frac{p}{p'} = \frac{q}{q'} = k \; ; \tag{29}$$

then $p = kp'$, $q = kq'$; and these values substituted in (8) give

$$mn\,(p' - q') = kp'q'\,(m - n) \; ; \tag{30}$$

hence by (22)

$$mn = kp'q' \; , \tag{31}$$

and by (29)

$$\frac{mn}{p'q'} = \frac{p}{p'} = \frac{q}{q'} \; ; \tag{32}$$

$\therefore$

$$mn = pq' = p'q \; . \tag{33}$$

$\therefore$ Segments intercepted by a chord and the circle described upon the opposite chord are reciprocally proportional to the segments intercepted by the given circle.

Now in Fig. 5 draw the chords $CC'$, $DD'$, and upon these also describe circles passing through $O$. We thus have an inscribed quadrilateral with a circle described upon every side and passing through the intersection of the diagonals as shown in Fig. 6.

Let the notation for the first pair of lines and circles be the same, and also let $OE_4 = r$, $OE_1 = r'$, $OE' = s$, $OE_4' = s'$. Then, since (33) is true under all conditions, we have

$$mn = rs' = r's \; . \tag{34}$$

However, this can be shown independently by a process just like the one used from (26) to (33), using (12) and (23) instead of (8) and (22).

In the same way using (18), (19), and (24) we can show that (33), (34) are true when $O$ is without the given circle.

Now combining (33) and (34) gives

$$mn = pq' = p'q = rs' = r's \; . \tag{35}$$

Since the product $mn$ is constant it follows that this equation is also true when $O$ is outside and the transversal does not cut the given circle. Thus the

formula in (35) is perfectly general and expresses the desired fundamental relation.
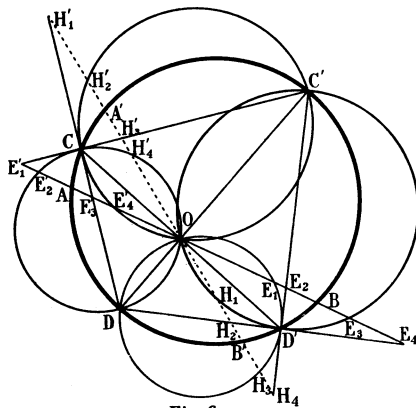


*Fig. 6.*

Next, to consider its consequences.    Every right line through $O$ cuts each of the sides of the quadrilateral, each of the four circles through $O$ in one other point and the given circle in two points (when $O$ is within it), so that eight of the ten factors in (35) are always real, and when one passes through infinity another must pass through zero.    Therefore when the transversal is parallel to a side of the quadrilateral it must be tangent to the circle described on the opposite side,* and when one point of intersection changes its direction from $O$ another changes also, and thus two lines change signs simultaneously.

As a particular case, when $O$ is the middle point of $AB$ it is readily seen from (24$a$) that (35) becomes

$$m^2 = n^2 = pp' = qq' = rr' = ss' . \qquad (36)$$

Restoring the values of the factors in (35) as shown on Fig. 6 we have

$$OA . OB = OE_1 . OE_1' = OE_2 . OE_2' = OE_3 . OE_3' = OE_4 . OE_4' . \quad (37)$$

It is now seen that the points $A$, $B$; $E_1$, $E_1'$; $E_2$, $E_2'$; $E_3$, $E_3'$; $E_4$, $E_4'$, form a system in involution of which the point $O$ is the centre.    Also that one pair of conjugate points lies on the given circle, and of every other pair one point lies on a side and its conjugate lies on the circle described on the opposite side.

Since $OA . OB$ is constant it follows that when $O$ is outside the given circle the locus of the foci is a circle whose centre is $O$ and whose radius is a mean proportional between the segments of either diagonal.

---

* This result is easily proved by means of arcs, and is cited simply as a check for this solution of the given problem.

As a result of this investigation the following general theorem can now be enunciated :—

THEOREM I. If a quadrilateral is inscribed in a given circle, and on each of its sides circles are described passing through the intersection of the diagonals, and any right line is also drawn through the intersection of the diagonals :

1. The product of the segments intercepted on this right line by the given circle, and the product of the segments intercepted by two opposite sides of the quadrilateral, are proportional to their respective $\begin{cases} \text{differences.} \\ \text{sums.} \end{cases}$ The same is true of the segments intercepted by the two pairs of opposite sides. See (9), (13), (20).

2. The $\begin{cases} \text{difference} \\ \text{sum} \end{cases}$ of the segments intercepted by the given circle is equal to the $\begin{cases} \text{difference} \\ \text{sum} \end{cases}$ of the segments intercepted by two circles described upon opposite sides of the quadrilateral. The same is true of the segments intercepted by the two pairs of opposite circles. See (22), (23), (24).

3. The segments intercepted by two opposite circles are divided proportionally by the sides upon which these circles are described, and are also proportional to the segments intercepted by the same sides. See (27), (28).

4. This right line (or transversal) is cut by the five circles and the four sides of the quadrilateral in ten points in involution of which the intersection of the diagonals is the centre. See (37).

5. When the intersection of the diagonals is outside of the given circle the locus of the foci* of the involution is a circle with the same centre as the involution and whose radius is a mean proportional between the segments of either diagonal.

6. When the segments intercepted by the given circle are equal, the segments intercepted by opposite sides, and the segments intercepted by opposite circles, are equal also ; and the segments made by the given circle are a mean proportional between the segments made by any side and the circle described upon that side. See (10), (24a), (36).

7. On every transversal (under conditions of 5) are determined five harmonic ranges, every range including the two foci ; for any two conjugate points of an involution and the two foci form a harmonic range (Smith's Conics, p. 59).

Let any other line $A'OB'$ (Fig. 6) be drawn giving the pairs of conjugate points, $H_1$, $H_1'$ ; $H_2$, $H_2'$ ; $H_3$, $H_3'$, etc.; then since

$$OA \cdot OB = OA' \cdot OB' ,$$
$$OE_1 \cdot OE_1' = OH_2 \cdot OH_2' = OE_4 \cdot OE_4', \text{ etc.}$$

* When the point is within there are no foci, for conjugate points are then on opposite sides of the centre.

8. ∴ Any pair of conjugate points on one transversal and any pair on any other transversal are concyclic.

Before extending this investigation any further it is necessary to call special attention to the fact that the term "quadrilateral" as here used does not mean the "complete quadrilateral" of modern geometry, which is considered as having three diagonals each of which is divided harmonically by the other two. On the contrary, any four points on the given circle are here supposed to determine in general three different inscribed quadrilaterals, as the three different pairs of intersecting lines are taken successively for diagonals. Thus the sides of these three inscribed quadrilaterals are all chords (not produced) of the given circle, and there are three points of "intersection of diagonals,"* one within and two without the given circle.

Inasmuch as all of the preceding formulas have been shown to be true under all possible conditions it is only necessary to say that Theorem I can be applied to all of these quadrilaterals.

With the aid of this theorem we can now prove very neatly a few additional theorems relating to concurrence and collinearity, and also discover some interesting relations connecting these three quadrilaterals and their systems of circles.

The reader is now referred to Fig. 7, which is necessarily a somewhat complicated diagram representing the three quadrilaterals with their systems of circles, and also the loci of the foci of two involutions, in all fifteen circles. The given circle and the system of quadrilaterals are drawn in heavy lines, and the four systems of circles in light lines ; the two focal loci are represented by dotted lines.

The three $D$-points are $O_1$, $O_2$, $O_3$. In the discussion of this figure it seems desirable to note, in the first place, that for each exterior $D$-point, in addition to the system of circles through that point, there are two circles belonging to each of the other systems which are coaxial with the given circle and having this $D$-point for their radical centre. Therefore these circles also cut any transversal drawn through this $D$-point in points which belong to the same involution, conjugate points lying on the same circle. A transversal may cut one, two, or three of these, so that there may now be as many as sixteen points and can not be fewer than eight in any such involution.

Bearing in mind, as has been stated, that if a point in any of these involutions lies on a side of the quadrilateral, its conjugate lies on the circle

---

* Hereafter, instead of "intersection of diagonals," we shall write "$D$-point." And four circles described on the sides of one quadrilateral and through its $D$-point, will be called a "system" of circles or "system." The two quadrilaterals whose $D$-points are without the given circle may be called "crossed quadrilaterals," since one pair of their opposite sides intersect within the given circle. Circles described upon opposite sides will be called "opposite circles."

described on the opposite side and *vice versa*, it is clear that the intersection
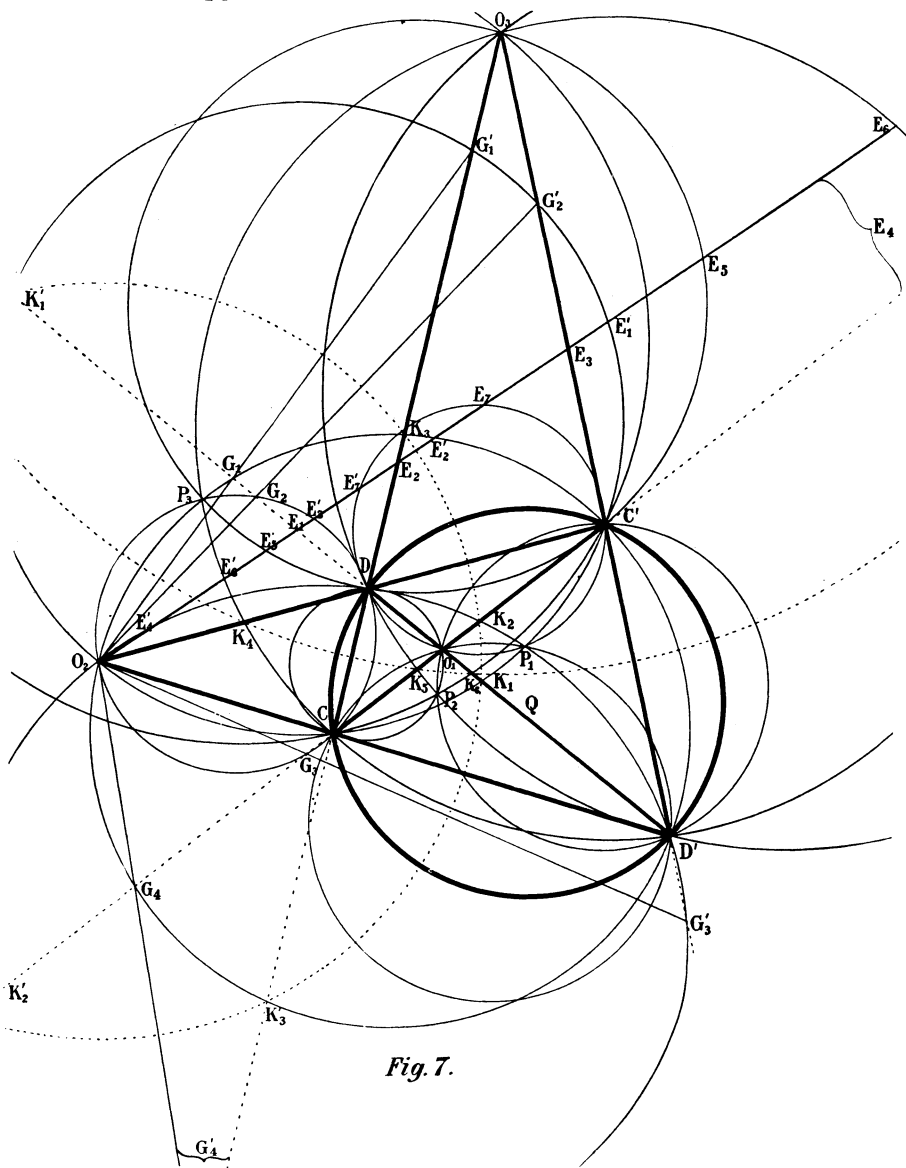


*Fig. 7.*

of a circle with the side opposite to that upon which it is described becomes
two coincident conjugate points and is therefore a focus.

(a) ∴ A circle described on a side of a crossed quadrilateral meets the

opposite side in a point whose distance from the $D$-point is a mean proportional between the segments of either diagonal.

For each exterior $D$-point there are six such points, some of which are shown at $K_1$, $K_1'$, $K_2$, $K_2'$, $K_3$, $K_3'$ for $O_2$, and $K_4$, $K_5$, $K_6$, for $O_3$.

If, in an involution, two points which are not conjugates coincide, the points which are conjugate to these respectively must also coincide. Thus, to the intersection of a side and an adjacent circle there corresponds the intersection of a circle and an adjacent side.

(b) $\therefore$ Two circles of the same system described upon adjacent sides meet the other two adjacent sides in two points which are collinear with the $D$-point of the system.

There are four such lines for each system, but only those for $O_2$ are drawn, viz. $O_2G_1G_1'$, $O_2G_2G_2'$, $O_2G_3G_3'$, $O_2G_4G_4'$.

Also, to the intersection of two opposite sides there corresponds the intersection of the circles upon those sides. This last point must be considered further. Let any right line be drawn through $O_2$ meeting the sides (counting from $O_2$) in $E_1$, $E_2$, $E_3$, $E_4$, and the circles on the respective opposite sides in $E_1'$, $E_2'$, $E_3'$, $E_4'$; let this line also meet the circle $O_3C'D$ in $E_5$, $E_5'$ and the circle $O_3CD$ in $E_6$, $E_6'$; conjugate points being designated by the same subscripts. Now suppose this line to revolve about $O_2$ until it passes through $O_3$, then $E_2$, $E_3$, $E_5$, $E_6$ coincide in $O_3$, and therefore their respective conjugates $E_2'$, $E_3'$, $E_5'$, $E_6'$ must coincide in some point as $P_3$. By supposing this line to revolve in the opposite direction till it passes through $O_1$ we can show that four different circles meet in a common point at $P_1$. This last statement will now be proved in a different way in order to secure an additional inference.

Using the former notation, and adhering to the convention that the segments intercepted by the given circle and also those intercepted by circles upon opposite sides shall have like signs when they extend in the same direction from the $D$-point, it will be remembered that (22) becomes general. It also follows that when the segments intercepted by two opposite circles coincide they must have the same sign, and (22) takes the form

$$m - n = p' + q'. \qquad (22')$$

Therefore, when a transversal passes through the intersection of two opposite circles, and hence also through the intersection of two opposite sides, we must have $p' = + q'$. Then by substituting in (22') we have

$$m - n = 2p', \text{ for } O_2; \qquad (38)$$

and

$$m + n = 2p', \text{ for } O_3. \qquad (39)$$

Thus in both cases the intersection of the opposite circles bisects the chord

intercepted by the given circle, and therefore all four circles meet in a common point. In precisely the same way it can be shown that the point $P_2$ is the middle of the chord intercepted on a line through $O_3O_1$, and also common to four circles.

(c) We thus have three other points* besides the three $D$-points through each of which four circles pass (no circle passing through more than one) and lying on the lines joining the $D$-points.

A second general theorem may now be stated:

THEOREM II. If a quadrilateral is inscribed in a given circle and on each of its sides circles are described passing through the $D$-point:

1. In a crossed quadrilateral three of these circles meet their respective opposite sides in points whose distance from the $D$-point is the mean proportional between the segments of either diagonal.         (a)

2. Two circles upon adjacent sides meet the other two adjacent sides in points which are collinear with the $D$-point.         (b)

3. Two circles described upon one pair of opposite sides and passing through the intersection of the other pair, and two circles described upon the second pair passing through the intersection of the first pair, all meet in a common point on the right line joining the intersections of the opposite sides, and bisect the chord of the given circle drawn through this point and the $D$-point.

For the sake of perspicuity in the diagram, let us now transfer the three $D$-points and these three $P$-points to Fig. 8. In this figure the triangle determined by the three $D$-points is drawn in heavy continuous lines and two sides produced to $P$-points; and the three $P$-points are joined with heavy dotted lines.

Now remembering that $O_1$, $O_2$, $O_3$ are each the centre of an involution in which each of the other two is the conjugate of a $P$-point, we have

$$O_1P_2 \cdot O_1O_3 = O_1P_1 \cdot O_1O_2 , \quad O_2O_1 \cdot O_2P_1 = O_2P_3 \cdot O_2O_3 ,$$

$$\text{and } O_3O_1 \cdot O_3P_2 = O_3P_3 \cdot O_3O_2 ; \quad (40)$$

hence the following groups of points are concyclic:

$$O_2, \ O_3, \ P_1, \ P_2 ; \ \ O_3, \ P_1, \ O_1, \ P_3 ; \ \ O_2, \ P_3, \ O_1, \ P_2 .$$

Draw these three circles in continuous lines and circumscribe a dotted circle about the triangle $O_1O_2O_3$; also draw the straight lines $O_2P_2$, $O_1P_3$, and $O_3P_1$. We now have a system of quadrilaterals having a side in the common line $O_2O_3$ and inscribed in each of these continuous circles and three circles through each of the points $O_1$, $O_2$, $O_3$.

---

* These three points are marked $P_1$, $P_2$, $P_3$, and when necessary to refer to them in a general way we shall write "$P$-point."

Consider the first of these quadrilaterals $O_2O_3P_1P_2$ with its interior $D$-point $O_1$. It is easily seen that each of the three circles through $O_1$ is described upon a side of this quadrilateral, and therefore any transversal
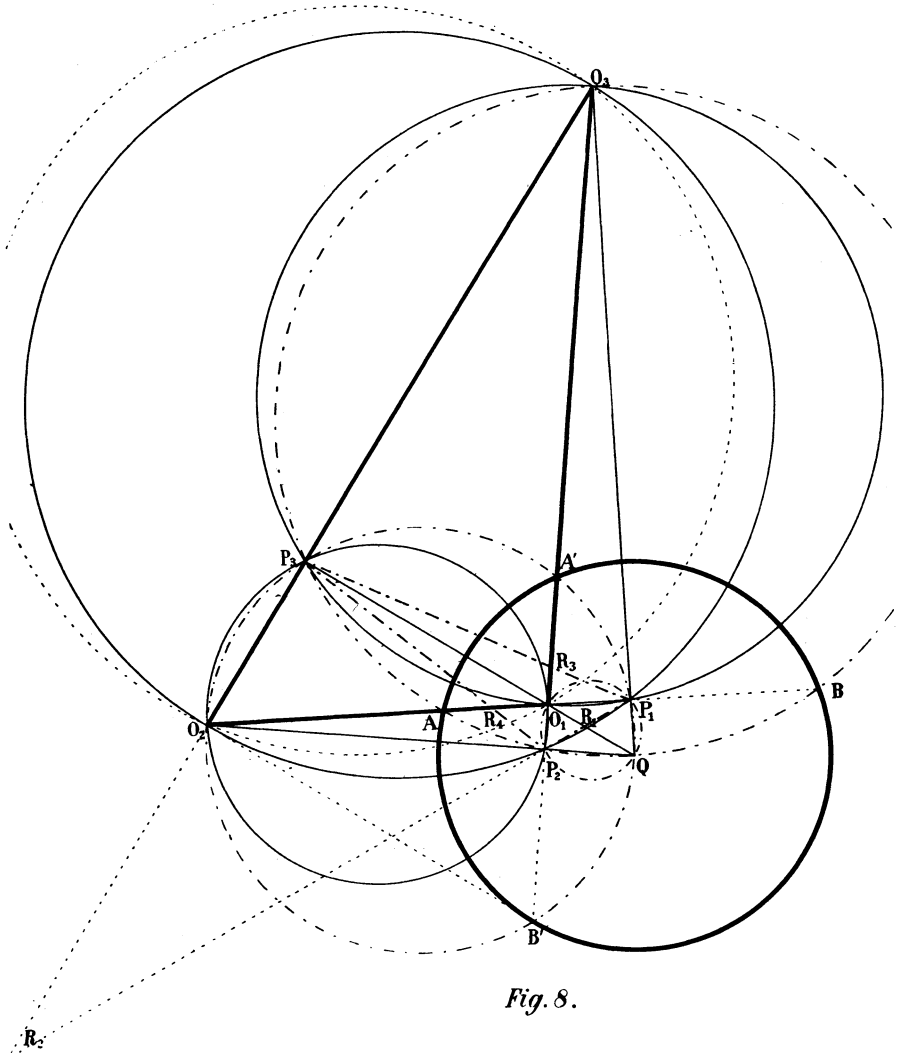


*Fig. 8.*

through $O_1$ is cut in involution by this system of circles and lines, and from the first equation of (40) it is evident that these points belong to the same involution we have been considering. The same thing can be shown to be true at the points $O_2$ and $O_3$, so that we may now add eight more points to every involution.

TRANSVERSALS, AND ITS CONSEQUENCES.

189

Furthermore it is readily perceived that

$$\angle\ O_2P_1P_2 = \angle\ O_2P_1P_3 = \angle\ O_3\ ,\quad \angle\ O_3P_2P_1 = \angle\ O_3P_2P_3 = \angle\ O_2\ .$$

∴ The triangles $O_1P_1P_2$, $O_2P_1P_3$, $O_3P_2P_3$ are each similar to $O_1O_2O_3$; the sides of the triangle $P_1P_2P_3$ are the *antiparallels* of the sides of the triangle $O_1O_2O_3$; and the sides of the triangle $O_1O_2O_3$ bisect the angles (exterior angle at $P_3$) of the triangle $P_1P_2P_3$.

Hence $O_1$ is the centre of the circle inscribed in the triangle $P_1P_2P_3$, and therefore $O_1P_3$ bisects the $\angle\ P_1P_3P_2$, and $O_1P_3$ is perpendicular to $O_2O_3$.

∴ $O_3O_1$ is the diameter of the circle $O_3P_3O_1P_1$,

$\qquad O_2O_1$ "         "         "         " $O_2P_3O_1P_2$,

and $\qquad O_2O_3$ "         "         "         " $O_3P_1P_2O_2$.

∴ A circle described on a line joining two $D$-points as a diameter passes through the two $P$-points that do not lie on that line.

As the lines $O_1P_3$, and $O_2P_3O_3$ bisect respectively the interior and exterior angle of the triangle $P_1P_2P_3$, the lines $P_3P_1$, $P_3O_1$, $P_3P_2$, $P_3O_2$ form a harmonic pencil intercepting the harmonic ranges $P_1R_1P_2R_2$, $P_1O_1R_4O_2$, $P_2O_1R_3O_3$.

For the same reason there are harmonic pencils at $P_1$ and $P_2$ intercepting the same range $O_3P_3O_2R_2$.

One ray of each of these pencils passes through each of the $D$-points. Also any line through a $D$-point is a transversal of all these pencils, save when it coincides with a ray.

∴ Every right line through a $D$-point is not only cut in involution, but is also cut in three harmonic ranges to which this $D$-point is common.

It has been proved that $O_3P_1$, $O_1P_3$, $O_2P_2$ are perpendicular to $O_2O_1$, $O_2O_3$, $O_1O_3$, respectively, and also, in (38), (39), that $P_1$, $P_2$ are the middle points of chords of the given circle which lie in these lines $O_2O_1$, $O_3O_1$.

∴ The three altitudes of the triangle $O_1O_2O_3$ meet in $Q$, which is the centre of the given circle.

Next describe a circle upon $O_2Q$ as a diameter. This circle, drawn in dots and dashes, passes through $P_1$, $P_3$, since $\angle\ O_2P_1Q = \angle\ O_2P_3Q =$ Rt. $\angle$. Also a circle on $O_3Q$ as a diameter passes through $P_2$, $P_3$; and a circle on $O_1Q$ as a diameter passes through $P_1$, $P_2$.

∴ As these circles are each described upon a side of the quadrilaterals now under consideration, it is evident that they and their corresponding sides give points in the same involution and that the centre of the given circle is a point in the involution on the lines drawn through it, its conjugates being respectively $P_1$, $P_2$, $P_3$.

Let us now consider the quadrilateral $P_3O_1O_3P_1P_3$. The circle upon $O_2Q$ as a diameter, being also described on $P_1P_3$, intersects the opposite side $O_3O_1$ in two points, as $A'$, $B'$, which lie on the locus of the foci, since both are coin-

cident conjugate points.   It also cuts the given circle in two points which are the points of contact of tangents through $O_2$; but these points of contact are also coincident conjugate points, and hence lie on the locus of the foci.   Therefore the two circles and the locus of the foci meet the line $O_3O_1$ in the same two points.

And since $O_2A'$, $O_2B'$ are tangents to the given circle, $O_2$ is the pole of $O_3O_1$ with respect to this circle.   In the same way it can be shown that $O_3$ is the pole of $O_2O_1$, and therefore $O_1$ is the pole of $O_2O_3$.

∴ The triangle of the $D$-points is self-polar with respect to the given circle; which is a well known theorem.

Again, it is easily seen that $Q$ is a common $D$-point for all the quadrilaterals that are inscribed in the circles which are described upon the sides of the self-polar triangle as diameters, and is therefore the radical centre of these circles.   Therefore, any right line through $Q$ is cut in involution by the sides of the self-polar triangle and by all these circles (save the one circumscribing $O_1O_2O_3$).   In these involutions if a point lies on a circle through $Q$ and a vertex of the self-polar triangle, its conjugate lies on the side opposite this vertex, other circles cutting in conjugate points.

Again, from the right triangle $QO_2B'$ it is evident that

$$QP_2 \cdot QO_2 = \overline{QB'}^2.$$

∴ The given circle is the locus of the foci of these involutions and cuts orthogonally the circles described upon the sides of the self-polar triangle.

These last results merit special attention and will be put in the form of a general theorem.

THEOREM III. If a triangle is self-polar with respect to a given circle, and circles are described on its sides as diameters, and also three other circles are drawn having for their diameters the right lines joining the centre of the given circle (intersection of altitudes) to the vertices of the self-polar triangle :

1. Every straight line drawn through the centre of the given circle is cut by these six circles and the sides of the self-polar triangle in an involution of which the centre of the given circle is the centre.

2. The given circle is the locus of the foci of the involution determined upon this variable line, and also cuts orthogonally the three circles described upon the sides of the self-polar triangle.

3. Every diameter of the given circle is divided harmonically, (1) by each of the circles upon the sides of the self-polar triangle (provided it cuts the diameter), (2) by each side of the self-polar triangle and its respective circle which passes through the opposite vertex of the triangle and the centre of the given circle.

[TO BE CONCLUDED.]